Contact L-graphs and their relation with planarity and chordality

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B_0-CPG graphs

- An undirected graph $G = (V, E)$ is called a VPG graph ([1]) if one can associate a path in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point.
- An undirected graph $G = (V, E)$ is then called a B_k-*VPG graph*, for some integer $k \geq 0$, if one can associate a path with at most k bends in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point.
- An undirected graph $G = (V, E)$ is said to be B_0-CPG if one can associate a horizontal or vertical path in a rectangular grid with each vertex, such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point without crossing each other and without sharing an edge of the grid.

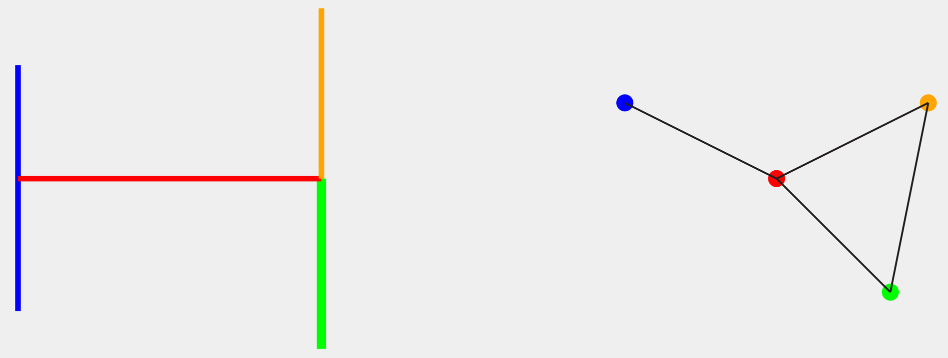

Figure: On the left, a B0-CPG representation of the graph on the right.

L-Contact graphs

- An ⌞-*graph* is a graph with a B_1-VPG representation such that all the paths in the representation have the shapes $\{|, -, ⌞\}$. We will say that the graph is an *strict* ⌞-*graph* if the paths only have the shape ⌞.
- An *(strict)* ⌞-*contact graph* is an (strict) ⌞-graph such that all the paths in the representation do not cross each other and do not share an edge of the grid.
- A representation of a strict ⌞-contact graph such that no path intersects another in a bend point will be called a *basic* representation.

Figure: Two representations of K_3 as a strict ⌞-contact graph.

Relation with planarity

- B_0-CPG $\subseteq$ ⌞-contact and there are non-planar B_0-CPG graphs.

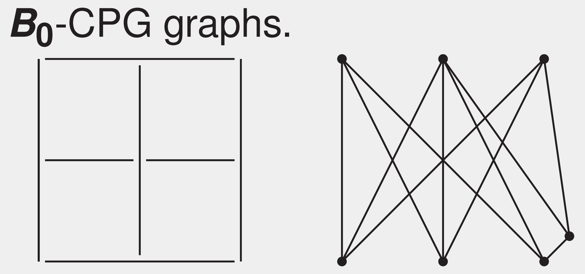

Figure: On the left, a B0-CPG representation of the non-planar graph on the right.

- As a consequence, there are non-planar ⌞-contact graphs.
- ⌞-contact $\subseteq$ B_1-CPG.

Theorem ([3])

For every $k \geq 0$ there is a planar graph G such that G is B_{k+1}-CPG but not B_k-CPG.

Theorem

If G is strict ⌞-contact then G is planar.

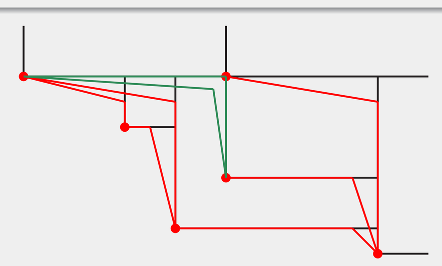

Figure: The planar representation obtained from the strict ⌞-contact representation of a graph.

Laman graph

- A *Laman graph* is a graph on n vertices such that, for all k, every k-vertex induced subgraph has at most $2k - 3$ edges, and such that the whole graph has exactly $2n - 3$ edges.
- An ⌞*-contact representation is a B_1-CPG representation which is strict and basic.
- an ⌞*-contact representation is maximal if every endpoint that is neither bottommost, topmost, leftmost, nor rightmost makes a contact, and there are at most three endpoints that do not make a contact.

Theorem ([4])

If a graph G has a maximal ⌞-contact representation in which each inner face contains the right angle of exactly one ⌞, then G is a planar Laman graph.*

- As a consequence, we have the following result.

Theorem

Every maximal strict ⌞-contact graph is a planar Laman graph.

Relation with chordality

Lemma

A clique in a strict ⌞-contact graph has size at most three.

Theorem

Let G be a chordal graph. G is strict ⌞-contact if and only if G is K_4-free. Moreover, G admits a basic representation.

Let $\mathcal{T}$ be the family of graphs defined as follows. $\mathcal{T}$ contains H_0 as well as all graphs constructed in the following way: start with a tree of maximum degree at most three and containing at least two vertices; this tree is called the *base tree*; add to every leaf v in the tree two copies of K_4 (sharing vertex v), and to every vertex w of degree 2 one copy of K_4 containing vertex w. Notice that all graphs in $\mathcal{T}$ are chordal.

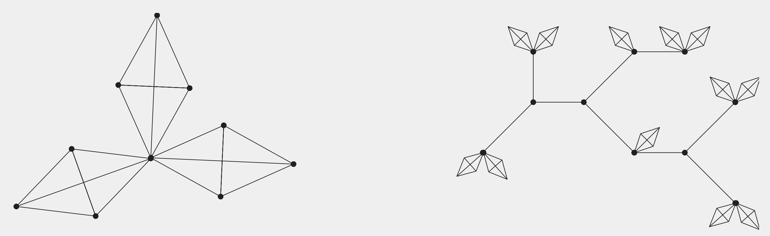

Figure: On the left the graph H_0. On the right a typical graph in $\mathcal{T}$.

Theorem ([2])

Let G be a chordal graph. Let $\mathcal{F} = \mathcal{T} \cup \{K_5$,diamond$\}$. Then, G is a B_0-CPG graph if and only if G is $\mathcal{F}$-free.

- It is immediate that ⌞-contact graphs are K_5-free and that B_0-CPG $\subseteq$ ⌞-contact.
- Following the same ideas as in the chordal B_0-CPG characterization, all the graphs in $\mathcal{T}$ are forbidden subgraphs.
- As a consequence, we have the following result concerning block graphs.

Theorem

Let G be a block graph. G is ⌞-contact if and only if G is B_0-CPG.

References

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